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# MELNIKOV-BASED OPEN-LOOP CONTROL OF ESCAPE FOR A CLASS OF NONLINEAR SYSTEMS

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## ABSTRACT

The performance of certain nonlinear stochastic systems is deemed acceptable if, during a specified time interval, the systems have sufficiently low probabilities of escape from a preferred region of phase space. We propose an open-loop control method for reducing these probabilities. The method is applicable to stochastic systems whose dissipation- and excitation-free counterparts have homoclinic or heteroclinic orbits. The Melnikov relative scale factors are system properties containing information on the frequencies of the random forcing spectral components that are most effective in inducing escapes. This information is useful in practice even if the dissipation and excitation terms are relatively large. An ideal open-loop control force applied to the system would be equal to the negative of a fraction of the exciting force from which the ineffective components have been filtered out. Limitations inherent in any practical control system make it impossible to achieve such an ideal control. Nevertheless, numerical simulations show that substantial advantages can be achieved in some cases by designing control systems that take into account the information contained in the Melnikov scale factors.

## INTRODUCTION

The performance of certain nonlinear stochastic systems is deemed acceptable if, during a specified time interval, the systems have sufficiently low probabilities of escape from a preferred region of phase space. For example, the motion of a ship subjected to wave loading may be modeled by an equation of motion with a nonlinear restoring term (see, e.g., Hsieh, Troesch and Shaw, 1994). Given a design sea state with a specified mean return period, a coordinate defining the behavior of the ship (e.g., its roll angle) must have an acceptably small probability of exit from the "safe" region of phase space.

However, if that design sea state is exceeded, the probability of exit from the safe region may become unacceptably large. That probability could be reduced -- that is, the system could be stabilized -- by resorting to a suitable control strategy.

We propose a Melnikov-based procedure aimed at achieving efficient stabilization by open-loop control. The proposed procedure is applicable to the wide class of multistable systems which have dissipation- and forcing-free counterparts possessing homoclinic or heteroclinic manifolds. Examples are the system just described, the rf-driven Josephson junction, the Duffing equation, as well as higher- or infinitely-dimensional systems (Holmes and Marsden, 1981; Wiggins and Holmes, 1987, Wiggins and Shaw, 1988; Allen et al, 1991; Zhang and Falzarano, 1994; Simiu and Frey, 1995; Simiu, 1995).

We review in the following section the theoretical basis of our procedure. Next, to test its effectiveness, we use numerical simulations for the paradigmatic case of a stochastichally excited Duffing equation. Finally, we discuss the results and present our conclusions.

## MELNIKOV PROCESSES AND EXITS FROM A WELL

For a class of dynamical systems described later in this section, the Melnikov approach is a technique providing necessary conditions for the occurrence of chaos -- and of exits from regions of phase space associated with potential wells. Originally it was considered to be applicable only to deterministic systems, including systems with quasiperiodic excitation (Beigie et al., 1991). However, the approach was recently extended to systems with stochastic excitation (Frey and Simiu, 1993). One remarkable result of this extension is that, under certain conditions, a motion can be both stochastic (i.e., induced by a realization of a stochastic process) and chaotic (i.e., sensitive to initial conditions). (See also Seki et al., 1993.)

Proofs used in Melnikov theory require that (i) the excitation terms be uniformly bounded and uniformly continuous, and (ii) the excitation and dissipation terms be asymptotically small. The first requirement was shown to be consistent with the application of Melnikov theory to systems excited by a wide class of random processes, including processes approximating as closely as desired broadband Gaussian noise (e.g., by using the Shinozuka representation -- see Shinozuka, 1971, Shinozuka and Deodatis, 1991), white noise, shot noise, and dichotomous noise (Frey and Simiu, 1993; Simiu and Hagwood, 1994; Frey and Simiu, 1995). It was also shown that the second requirement can be considerably relaxed in practice, that is, even if the excitation and dissipation terms are relatively large, Melnikov theory can be helpful in the search for chaos (Guckenheimer and Holmes, 1983; Moon, 1987), and for selecting appropriate control force frequencies and amplitudes to increase mean exit times (Franaszek and Simiu, 1995).

For definiteness we consider the equation

$$\ddot{z} = -V'(z) + \epsilon[\gamma G(t) - \beta\dot{z})] \qquad (1)$$

where $\beta,\gamma$ are constants, $\beta>0$, and $V(z)$ is a potential function. We assume that: (i) the unperturbed system ($\epsilon=0$) is integrable; (ii) $V(z)$ has the shape of a multiple well so that the unperturbed system has a center at the bottom of each well and a saddle point at the top of the barrier between two adjacent wells, and that the stable and unstable manifolds emanating from the saddle point of the unperturbed system are homoclinic or heteroclinic; and (iii) $\epsilon$ is sufficiently (though not asymptotically) small. Finally, we assume $G(t)$ is a random process. As a typical example belonging to the class of systems just described we consider in this note the Duffing equation, which has potential

$$V(z) = z^4/4 - z^2/2, \qquad (2)$$

homoclinic orbits with coordinates

$$z_s(t)=(2)^{1/2}\text{sech}(t); \quad \dot{z}_s(t)=(2)^{1/2}\text{sech}(t)\tanh(t) \qquad (3a, 3b)$$

and a modulus of the Fourier transform of the function $h(t)\equiv\dot{z}_s(-t)$

$$S(\omega)=(2)^{1/2}\pi\omega\,\text{sech}(\pi\omega/2). \qquad (4)$$

The function $S(\omega)$ is referred to as the Melnikov relative scale factor (Beigie et al., 1992). We also note for later use that

$$c \equiv \int_{-\infty}^{\infty} \dot{z}_s^2(\tau)d\tau = 4/3 \qquad (5)$$

As indicated earlier, the Melnikov approach is also applicable to systems of higher order, including spatially-extended dynamical systems and systems with multiplicative noise. For this reason this is also true of the arguments on Melnikov-based open-loop control developed in this note.

Associated with Eq. 1 is a Melnikov process with the expression

$$M(t) = -\beta c + \gamma \int_{-\infty}^{\infty} h(\tau)G(t-\tau)d\tau. \qquad (6)$$

Any realization of the Melnikov process represents the distance between the stable and unstable manifolds of Eq. 1 ($\epsilon\neq0$) corresponding to a realization of the random process $G(t)$.

The mean zero upcrossing time, $\tau_M$, of the Melnikov process induced by the excitation is a measure of the mean time of exit from a well, $\tau_e$ (Simiu, 1995). For any given system, increasing $\tau_M$ -- and $\tau_e$ -- by using an open-loop control approach can be achieved by adding to the excitation $\epsilon\gamma G(t)$ a control force $\epsilon\gamma_c G_c(t)$, where $\gamma_c$ has the same sign as $\gamma$.

A trivial choice of the open-loop control force would be $G_c(t)\equiv-G(t)$. Since the net excitation would then be smaller than $\epsilon\gamma G(t)$, it is seen from Eq. 6 that the addition of the control force would decrease the ordinates of the Melnikov process and increase its mean zero upcrossing time. It is clear that it would also result in an increased mean exit time for the system.

For this trivial choice of the control force, the ratio between the average power of the exciting force and the average power of the control force is $Q=\gamma^2/\gamma_c^2$. We seek to use the information contained in the Melnikov relative scale factor $S(\omega)$ to obtain open-loop control forces that would achieve results comparable to those achieved by the trivial control, but with considerably more effectiveness, that is, with an increased power ratio Q.

From Eq. 6 it follows that the spectral density of the Melnikov process for the uncontrolled system is

$$2\pi\Psi_M(\omega)=S^2(\omega)[2\pi\Psi(\omega)] \qquad (7)$$

where $S(\omega)$ is the modulus of the Fourier transform of $h(t)$, and $2\pi\Psi(\omega)$ is the spectral density of the random process $G(t)$. To illustrate Eq. 7, we consider the Duffing equation, for which $S(\omega)$ is given by Eq. 4, and the process $G(t)$ with spectral density (Fig. 1)

$$2\pi\Psi(\omega)=\begin{cases} 0.03990\ell n(\omega)+0.12829 & 0.04\leq\omega\leq 0.4 \\ 0.05755\ell n(\omega)+0.14493 & 0.4\leq\omega\leq 1.2 \\ -0.38301[\ell n(\omega)]^2+1.06192\ell n(\omega)-0.02941 & 1.2\leq\omega\leq 15.4 \end{cases} \qquad (8)$$

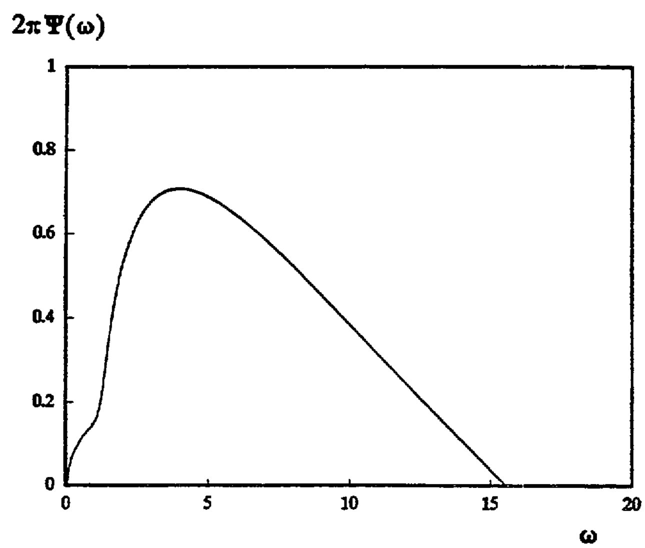


Fig. 1. Spectral density of excitation.

To a first approximation this spectrum is representative of low-frequency fluctuations of the horizontal wind speed (Van der Hoven, 1957). The functions $S^2(\omega)$ and $2\pi\Psi(\omega)S^2(\omega)$ are represented in Figs. 2a and 2b, respectively. Figs. 1 and 2 show that, owing to the shape of $S(\omega)$ -- which plays the role of an admittance function -- only part of the frequency components of the excitation $G(t)$ contribute significantly to the spectral density of the uncontrolled system's Melnikov process (for example, components with frequencies $\omega>4$ are suppressed; components with frequencies $2.5<\omega<4$ are very strongly reduced).

The following approach appears reasonable. Instead of $G_c(t)\equiv-G(t)$, it would be more efficient to apply a control force obtained from the function $-G(t)$ by filtering out from this function those frequency components that do not contribute significantly to the spectral density $\Psi_M(\omega)$. The advantage of this approach over the trivial approach $G_c(t)\equiv-G(t)$ is that, in general, it would reduce significantly the power needed for the system's control, while achieving a comparable reduction of (i) the ordinates -- and the mean zero upcrossing time -- of the controlled system's Melnikov process and, hence, (ii) the system's mean exit time.

Like its trivial counterpart, the approach just described is not feasible owing to practical limitations on the operation of the control system. These limitations entail non-zero time lags between sensing of a signal and the actuator response. In addition, the practical filters may entail other inefficiencies, although the opposite can be the case if the filter design is judicious. In the next section we present results of numerical simulations aimed at illustrating the potential of our approach, modified to account for practical control system limitations.

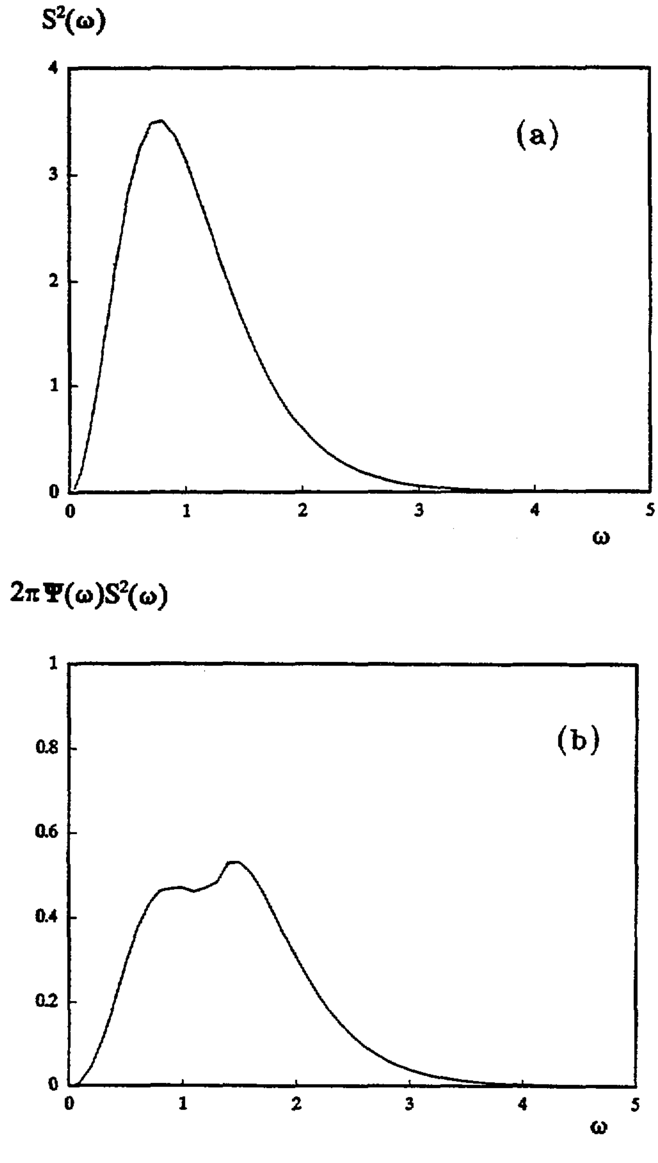


Fig. 2. (a) Square of Melnikov relative scale factor; (b) Spectral density of Melnikov process.

## NUMERICAL SIMULATIONS

We considered the Duffing equation (Eqs. 1 and 2), $\epsilon=0.1$ and $\beta=0.45$. We examined two cases: (1) $2\pi\Psi(\omega)$ is given by Eq. 8; (2) $2\pi\Psi(\omega)=2\pi/5$ for $0<\omega<5$ and $2\pi\Psi(\omega)=0$ otherwise.

We first estimated by numerical simulation the mean exit rate for the uncontrolled system. We then estimated the mean exit rate for the system with control forces. We considered four types of control force. The first type of control force, denoted by (a) and referred to here as trivial control force, has the expression $\epsilon\gamma_{ct}G(t-\tau_o)$. The second type, denoted by (b) and referred to here as ideal control force, was obtained by passing the function $-\epsilon\gamma_{ci}G(t-\tau_o)$ through an ideal filter that suppresses all the Fourier components for $0\leq\omega<\omega_1$ and $\omega>\omega_2$, and leaves the other components unchanged. The third type, denoted by (c) and referred to here as practical-filter control force, was obtained by passing the function $-\epsilon\gamma_{cp}G(t-\tau_o)$ through the filter with impulse response represented in Fig. 3 (a=0.1, b=2.25). The fourth type, denoted by (d) and referred to here as modified practical-filter control force, was obtained by passing

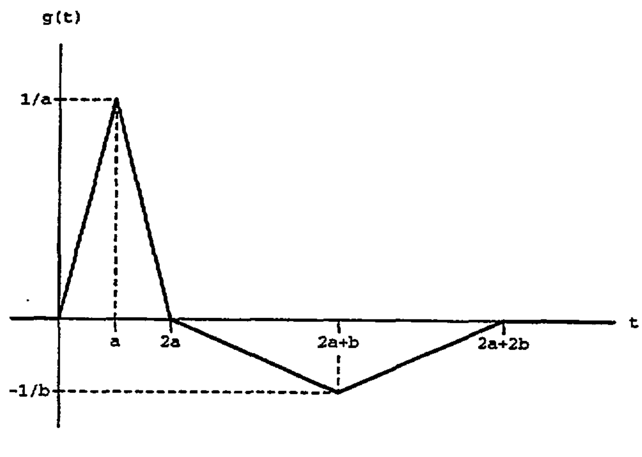


Fig. 3. Impulse response of a two-parameter filter with initial response and recoil.

the function $-\epsilon\gamma_{cm}G(t-\tau_o)$ through the filter with impulse response represented in Fig. 3 (a=0.1, b=2.25), and then suppressing from the output all Fourier components for $0\le\omega<\omega_1$ and $\omega>\omega_2$ while leaving the other components unchanged.

The time lag was assumed to be $\tau_o=0.1$. The frequencies $\omega_1$ and $\omega_2$ defining the intervals over which inefficient components were suppressed in (b) and (d) were chosen by examining the spectra of the Melnikov processes. The choices were $\omega_1=0.3$, $\omega_2=2.5$ (case 1) and $\omega_1=0.3$, $\omega_2=2.0$ (case 2).

The values of the coefficients $\gamma_{ct}$ and $\gamma_{ci}$ were chosen so that the control forces (a) and (b) have the same average power. (To within a constant, the average power is simply the variance of the control force.) Similarly, the values of $\gamma_{cp}$ and $\gamma_{cm}$ were chosen so that the control forces (c) and (d) have the same power. We assumed $\gamma_{ci}=0.5$ and $\gamma_{cm}=0.5$. The equal average power criterion yielded $\gamma_{ct}=0.195$ and $\gamma_{cp}=0.167$ for case 1, and $\gamma_{ct}=0.347$ and $\gamma_{cp}=0.292$ for case 2.

The real and imaginary parts of the transfer function of the filter of Fig. 3 are

$$R(v)=r^2(a\omega/2)\cos(a\omega)-r^2(a\omega/2)\cos(2a+b)\omega \qquad (7a)$$

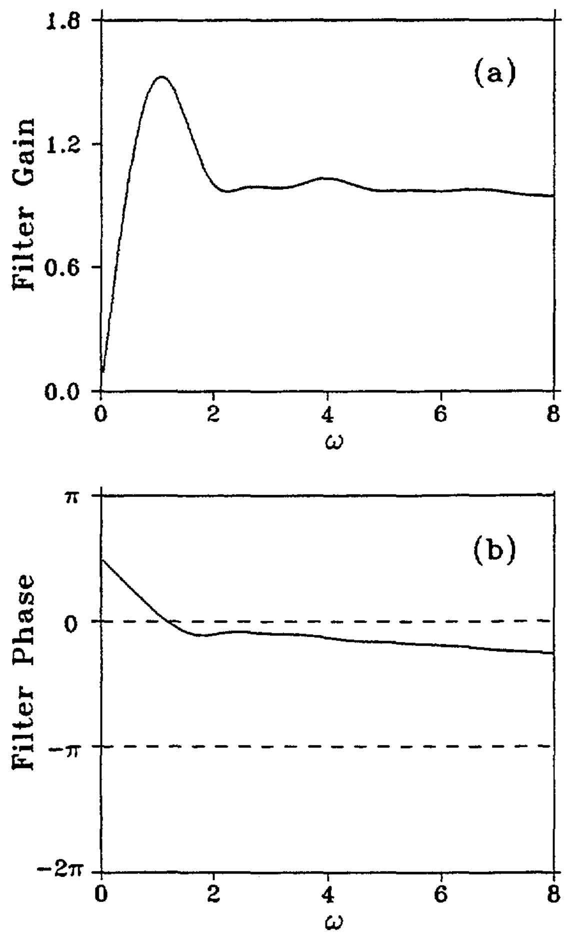


Fig. 4. (a) Gain and (b) phase angle for filter of Fig. 3 with a=0.1 and b=2.75.

$$I(\omega)=-r^2(a\omega/2)\sin(a\omega)+r^2(b\omega/2)\sin(2a+b)\omega \qquad (7b)$$

where $r(x)=\sin(x)/x$. Equations 7 were obtained from expressions available, e.g., in Papoulis (1962).) We show in Fig. 4 the dependence on frequency of the filter gain and phase.

The numerical simulations were performed by the adaptive step-size Runge-Kutta method. The realizations of the excitation process G(t) were simulated by sums of twenty-five sine and cosine terms with equally spaced frequencies and amplitudes distributed normally with zero mean and variance $2\pi\Psi(\omega)\Delta\omega$, where $\Delta\omega$ is the frequency increment (Rice, 1954). For each realization the initial points were chosen randomly and the trajectories were integrated for a time interval $T_{tot}=1000T$, where $T=2\pi/\omega_{max}$ and $\omega_{max}$ is the maximum energy-containing frequency of the spectrum of G(t). The number of zero

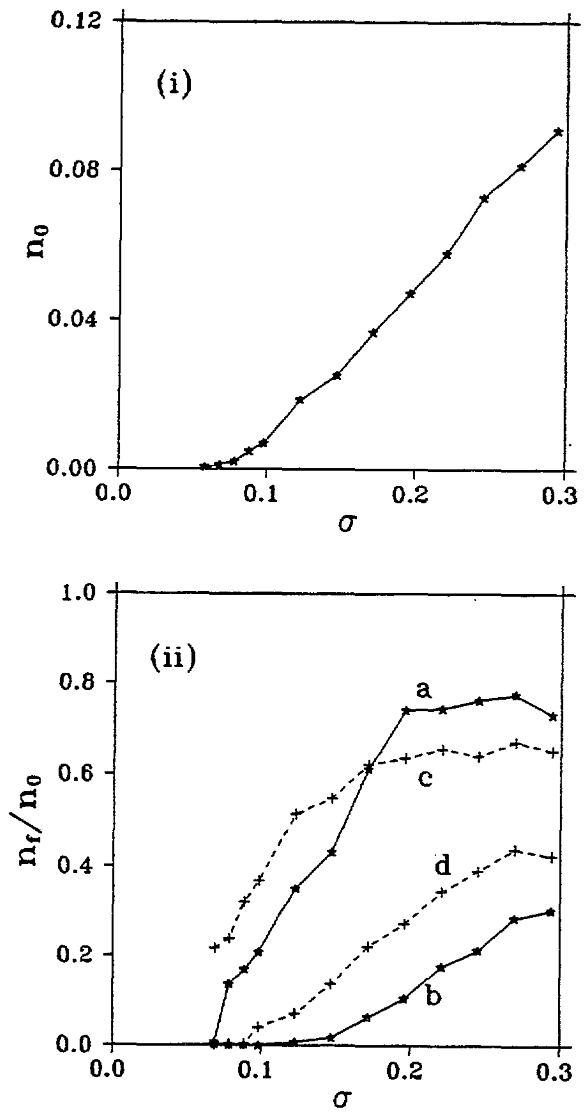


Fig. 5. Case 1: (i) Escape rate $n_o$ for uncontrolled oscillator subjected to noise $\sigma=\epsilon\gamma$; (ii) Ratio $n_f/n_o$ between escape rate of controlled and uncontrolled system. Curves (a) (b), (c), (d) are described in the text; $\sigma=\epsilon\gamma$.

crossings was counted for each of a total of 800 realizations. A similar procedure was applied to the controlled system.

The results are shown for cases 1 and 2 in Figs. 5 and 6, where $\sigma \equiv \epsilon\gamma$.

## DISCUSSION

The benefit that may in principle be derived from the knowledge of the Melnikov properties of the system can be assessed from a comparison of mean exit rates induced by a control force modified to take advantage of that knowledge, on the one hand, and by the unmodified counterpart of that force, on the other. Recall that we considered two types of loading: case 1, corresponding to the excitation spectrum of Eq. 8 (Fig. 2b), and case 2, corresponding to a uniform excitation spectrum for $0<\omega<5$, and a vanishing spectrum for $\omega \geq 5$.

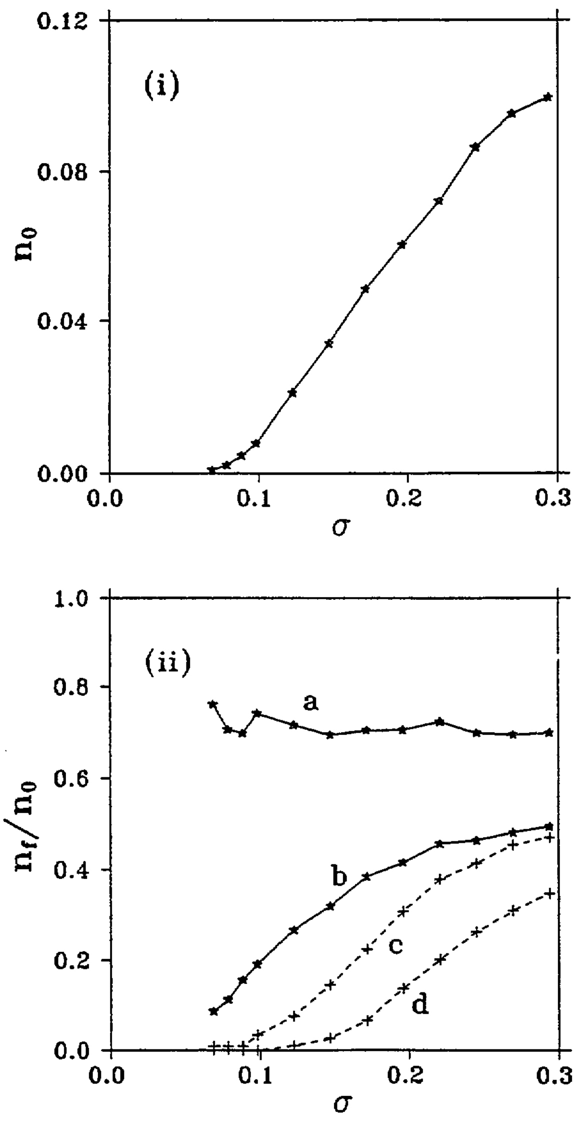


Fig. 6. Case 2: same legend as for Fig. 5.

Recall also that in Figs. 5 and 6 curves (a) correspond to trivial control forces, that is, forces equal to $0.5\epsilon\gamma G(t-0.1)$, where $\epsilon\gamma G(t)$ denotes the excitation force. Curves (b) correspond to forces obtained from the trivial control forces by eliminating inefficient components, and then amplified so that they have the same average power as the trivial forces. The use of knowledge inherent in the Melnikov properties of the system is seen to be useful in both cases 1 and 2. Note that the benefit due to the Melnikov-based control force is significantly stronger in case 1. This is due to the respective relative shapes of the Melnikov scale factor and the spectral density of the excitation. Curves (c) and (d) are the counterparts of curves (a) and (b) for the practical-filter control forces. Comments similar to those made for curves (a) and (b) are applicable for curves (c) and (d). The results of Figs. 5 and 6 also show that the benefits that accrue from the use of knowledge of Melnikov properties can also depend significantly upon the type of filter.

## CONCLUSIONS

A Melnikov-based open-loop approach to the control of a wide class of nonlinear stochastic systems was proposed. The aim of the proposed approach is to achieve a relatively efficient stabilization of the system. Exploratory numerical simulations suggested that the information contained in the Melnikov relative scale factors can help to achieve this objective. It is emphasized that our calculations fully accounted for nonlinearity of the system at hand.

The degree to which an efficient Melnikov-based open-loop control can be accomplished in practice depends upon the system under consideration (i.e., upon its Melnikov scale factors), the spectral density of the excitation, and the quality of the filter design. The intent of this note is not to study the filter problem in the context of Melnikov-based open-loop control. Rather, it is to draw the attention of control specialists to the approach proposed herein, in the belief that, whether used singly or as a component of a more complex control strategy, it may become a useful addition to the current body of nonlinear control theory and practice.


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